P54 - 55

3\. (a) Since $z_0 \neq 0$, $\lim_{z \to z_0} z^n = (\lim_{z \to z_0} z)^n = z_0^n \neq 0$.

Then $\lim_{z \to z_0} \frac{1}{z^n} = \frac{1}{\lim_{z \to z_0} z^n} = \frac{1}{z_0^n}$

(b) $\lim_{z \to i} \frac{iz^3 - 1}{z + i} = \frac{i^4 - 1}{i + i} = \frac{1-1}{2i} = 0$

(c) Since $\lim_{z \to z_0} Q(z) = Q(z_0) \neq 0$, $\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z \to z_0} P(z)}{\lim_{z \to z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}$

$\square$

5\. When $z = x$ with $x \in \mathbb{R}$,

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x}{x}\right)^2 = 1.$$

When $z = iy$ with $y \in \mathbb{R}$,

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{iy}{-iy}\right)^2 = (-1)^2 = 1.$$

When $z = x + ix$ with $x \in \mathbb{R}$,

$$f(z) = \left(\frac{z}{\bar{z}}\right) = \left(\frac{x+ix}{x-ix}\right)^2 = \left(\frac{1+i}{1-i}\right)^2 = \frac{(1+i)^2}{(1-i)^2} = \frac{2i}{-2i} = -1$$

Therefore $\lim_{\substack{z \to z_0 \\ \text{along } z = x}} f(z) = 1$ and $\lim_{\substack{z \to z_0 \\ \text{along } z = x+ix}} f(z) = -1$.

Hence, $\lim_{z \to z_0} f(z)$ D. N. E.

$\square$

P61

2. (a) $f'(z) = 6z - 2$

(b) $f'(z) = 5(2z^2+i)^4(4z) = 20z(2z^2+i)^4$

(c) $f'(z) = \dfrac{2z+1-2(z-1)}{(2z+1)^2} = \dfrac{3}{(2z+1)^2}$

(d) $f'(z) = \dfrac{4(1+z^2)^3 \cdot 2z \cdot z^2 - 2z(1+z^2)^4}{z^4}$

$= \dfrac{2(1+z^2)^3}{z^3}\left(4z^2 - (1+z^2)\right)$

$= \dfrac{2(1+z^2)^3(3z^2-1)}{z^3}$

$\square$

3. (a) $P(z)' = (a_0 + a_1 z + \cdots + a_n z^n)'$

$= (a_0)' + (a_1 z)' + \cdots + (a_n z^n)'$

$= 0 + a_1 + \cdots + n a_n z^{n-1}$

$= a_1 + a_2 z + \cdots + n a_n z^{n-1}$

(b) Claim: $P^{(k)}(z) = k!\, a_k + \dfrac{(k+1)!}{1!} a_{k+1} z + \cdots + \dfrac{n!}{(n-k)!} a_n z^{n-k}$,

for any $k = 1, \cdots, n$

Pf : · By (a), it is true when $k=1$.

· Suppose it is true for $k=m$.

· When $k=m+1$,

$$P^{(k)}(z) = P^{(m+1)}(z) = \left(P^{(m)}(z)\right)'$$

$$= \left(m!a_m + \frac{(m+1)!}{1!}a_{m+1}z + \cdots + \frac{n!}{(n-m)!}a_n z^{n-m}\right)'$$

$$= (m+1)!a_{m+1} + \cdots + \frac{n!(n-m)}{(n-m)!}a_n z^{n-m-1}$$

$$= (m+1)!a_{m+1} + \cdots + \frac{n!}{(n-m-1)!}a_n z^{n-m-1}$$

By M.I., it is true for any $k=1,\cdots,n$.

Take $z=0$, we are done. □

4\. Note that $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

$$g'(z_0) = \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$

Then $$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}}$$

$$= \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)\left(\frac{z-z_0}{g(z)-g(z_0)}\right) \quad \left(\text{Since } g'(z_0)\neq 0\right)$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)}$$

$$= \lim_{z \to z_0} \frac{f(z)}{g(z)} \qquad (\text{since } f(z_0) = g(z_0) = 0)$$

$\square$.

8. (a) Pick any z_0. Write $z = x + iy$, $z_0 = x_0 + iy_0$

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{x - x_0}{(x - x_0) + i(y - y_0)}$$

$$\lim_{\substack{z \to z_0 \\ \text{along } z = x + iy_0}} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

$$\lim_{\substack{z \to z_0 \\ \text{along } z = x_0 + iy}} \frac{f(z) - f(z_0)}{z - z_0} = 0 \quad \text{since } f(z) \equiv f(z_0),\ \forall\, z = x_0 + iy.$$

Hence, $f'(z_0)$ D.N.E.

(b) Pick any z_0. Write $z = x+iy$, $z_0 = x_0+iy_0$

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{y-y_0}{(x-x_0)+i(y-y_0)}$$

$$\lim_{\substack{z\to z_0 \\ \text{along } z=x_0+iy}} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{y\to y_0} \frac{y-y_0}{i(y-y_0)} = \frac{1}{i}$$

$$\lim_{\substack{z\to z_0 \\ \text{along } z=x+iy_0}} \frac{f(z)-f(z_0)}{z-z_0} = 0 \quad \text{since } f(z) \equiv f(z_0),\ \forall\ z = x+iy_0$$

Hence, $f'(z_0)$ D.N.E.

$\square$

P 70 - 71

1. (a) $f(z) = \bar{z} = x-iy$

$u_x = 1$, $v_y = -1$

Since $u_x \neq v_y$, $f'(z)$ D.N.E.

(b) $f(z) = z - \bar{z} = (x+iy)-(x-iy) = 2iy$

$u_x = 0$, $v_y = 2$

Since $u_x \neq v_y$, $f'(z)$ D.N.E.

(c) $f(z) = 2x + ixy^2$

$u_x = 2$, $u_y = 0$, $v_x = y^2$, $v_y = 2xy$.

When $y \neq 0$, $u_y \neq -v_x$.

When $y = 0$, $u_x \neq v_y$.

Hence, $f'(z)$ D.N.E , $\forall z$.

(d) $f(z) = e^x e^{-iy} = e^x(\cos y - i\sin y)$

$= e^x\cos y - i(e^x \sin y)$.

$u_x = e^x\cos y$, $u_y = -e^x\sin y$, $v_x = -e^x\sin y$, $v_y = -e^x\cos y$.

When $y \neq k\pi$, $\forall k \in \mathbb{Z}$, $u_y \neq -v_x$.

When $y = k\pi$ for some $k \in \mathbb{Z}$, $u_x \neq v_y$.

Hence , $f'(z)$ D.N.E. , $\forall z$

$\square$

2 (a) $f(z) = iz + 2 = 2 - y + ix$

$u_x = 0$, $u_y = -1$, $v_x = 1$, $v_y = 0$.

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$f'(z) = u_x + iv_x = i$.

$f''(z) = 0$.

(b) $f(z) = e^{-x} e^{-iy} = e^{-x}\cos y + i(-e^{-x}\sin y)$

$$\begin{cases} u_x = v_y = -e^{-x}\cos y \\ v_x = -u_y = e^{-x}\sin y. \end{cases}$$

$$f'(z) = -e^{-x}\cos y + i e^{-x}\sin y = -f(z).$$

Then $f''(z) = (-f(z))' = -f'(z) = -(-f(z)) = f(z)$

(c) $f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$

$$\begin{cases} u_x = v_y = 3(x^2-y^2) \\ v_x = -u_y = 6xy. \end{cases}$$

$$f'(z) = 3(x^2-y^2) + i\,6xy.$$

$$\begin{cases} \tilde{u}_x = \tilde{v}_y = 6x \\ \tilde{v}_x = -\tilde{u}_y = 6y. \end{cases}$$

$$f''(z) = 6x + 6iy = 6z.$$

(d) $f(z) = \cos x\cosh y - i\sin x\sinh y$

$$\begin{cases} u_x = v_y = -\sin x\cosh y \\ v_x = -u_y = -\cos x\sinh y \end{cases}$$

$$f'(z) = -\sin x\cosh y - i\cos x\sinh y$$

$$\begin{cases} \tilde{u}_x = \tilde{v}_y = -\cos x \cosh y \\ \tilde{v}_x = -\tilde{u}_y = \sin x \sinh y \end{cases}$$

$$f''(z) = -\cos x \cosh y + i \sin x \sinh y = -f(z)$$

$\square$

3(a) $f(z) = \dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$, $(z \neq 0)$

$u_x = v_y = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

$u_y = \dfrac{2xy}{(x^2+y^2)^2}$

$v_x = \dfrac{-2xy}{(x^2+y^2)^2}$

Then $f'(z) = \dfrac{-x^2+y^2-i(2xy)}{(x^2+y^2)^2} = -\dfrac{1}{z^2}$ $(z \neq 0)$

(b) $f(z) = x^2 + iy^2$

$u_x = 2x$

$v_y = 2y$

$u_y = -v_x = 0$.

$f'(z) = 2x$ when $x = y$, i.e. $z = x + ix$.

$f'(z)$ D.N.E. when $x \neq y$.

(c) $f(z) = z \operatorname{Im} z = (x+iy)y = xy + iy^2$

$$u_x = y \qquad u_y = x \qquad v_x = 0 \qquad v_y = 2y$$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \text{ iff } \begin{cases} x = 0 \\ y = 0. \end{cases}$$

Hence $f'(z) = 0$ if $z = 0$

$f'(z)$ D.N.E. if $z \neq 0$.

$\square$

7(a) From 6, $f'(z_0) = e^{-i\theta_0}(u_r + iv_r)$.

By CR equation (polar version),

$$\begin{cases} r_0 u_r = v_\theta \\ r_0 v_r = -u_\theta. \end{cases}$$

Then $f'(z_0) = e^{-i\theta_0}\left(\frac{v_\theta}{r_0} - i\frac{u_\theta}{r_0}\right)$

$$= \frac{i}{r_0 e^{i\theta_0}}(u_\theta + iv_\theta)$$

$$= \frac{-i}{z_0}(u_\theta + iv_\theta)$$

(b) $f(z) = \frac{1}{z} = re^{-i\theta} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$

$$\begin{cases} u_\theta = -rv_r = \dfrac{-\sin\theta}{r} \\ v_\theta = ru_r = \dfrac{-\cos\theta}{r} \end{cases}$$

$$f'(z) = \frac{-i}{z}\left(\frac{-\sin\theta}{r} - i\frac{\cos\theta}{r}\right)$$

$$= \frac{-1}{z}\left(\frac{\cos\theta}{r} - i\frac{\sin\theta}{r}\right)$$

$$= -\frac{1}{z}\left(\frac{1}{re^{i\theta}}\right) = -\frac{1}{z^2}$$

$\square$